METHOD OF PROBABILITY ANALYSIS OF STATIC STOCHASTIC TEMPERATURE FIELDS IN TECHNICAL OBJECTS

A. G. Madera

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An efficient method is proposed for calculating the mathematical expectations, the correlation matrix, and the variances of static stochastic temperature fields for complicated technical objects with random parameters and under random conditions. The technical systems are modelled by equivalent electrical circuits with random elements. The method is exact and much more efficient than the Monte Carlo method.

Introduction. The temperature fields of technical objects (radioelectronic and microelectronic apparatus, heat exchangers, electric machines, etc.) under real operating conditions are stochastic.

The temperature fields of technical objects are stochastic because the parameters and characteristics of the objects, such as the heat-emission intensities, the sizes of the structural elements, the thermophysical characteristics of the materials, the gapwidths between structural components, the temperature of the surrounding medium or coolant, and the heat-transfer coefficients, are themselves random. The randomness of the parameters and the characteristics of technical objects is in turn a consequence of the technological statistical variations in the fabrication of the objects, the uncontrollability of random factors in the construction, and fluctuations of the external operating conditions.

All existing methods of analysis of stochastic temperature fields in complicated technical objects (with the exception of the simplest cases, which can be studied analytically) are based on (with the exception of the simplest cases, which can be studied analytically) are based on the method of statistical tests (Monte Carlo method) [1-3]. The Monte Carlo method, being universal, has a number of fundamental drawbacks which preclude its use for many problems [4, 5]: the number of tests is large, so that in practice the method is useful only for solution accuracy higher than 15-20%; the error of the method is established only with some probability; the pseudorandom numbers generated by different devices in the computer are only approximately independent and uniformly distributed in the interval [0, 1]. For these reasons methods different from the Monte Carlo method and not having its drawbacks must be developed for analyzing stochastic temperature fields.

The present paper is devoted to such a method. Our method makes it possible to calculate the probabilistic characteristics (mathematical expectations, variances, and correlation matrix) of static stochastic temperatures at different points of a technical object. The method is exact and the probabilitic characteristics are obtained in analytical form. All parameters and characteristics determining the temperature field of the technical object are random and can conform to any truncated distribution functions.

Thermal and Mathematical Models. The thermal model of the technical object is represented in the form of a thermal network of isothermal bodies (elements of the technical object), which exchange heat with one another and with the surrounding medium, and heat sources and sinks [6].

The mathematical model of the heat-transfer processes occurring in the thermal model is a system of stochastic linear equations which corresponds to the assumption that the intensities of heat emission, the thermophysical characteristics of the materials, and the heat-transfer coefficients are independent of the temperature.

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The equations of the stochastic mathematical model are automatically constructed by the topological matrix method of nodal potentials for the graph of the thermal network and have the form [7]

$$A\hat{G}A^{T}\hat{\bar{t}} = A\hat{\bar{P}} - A\hat{G}\hat{\bar{t}}_{c}, \tag{1}$$

where $\hat{t} = (\hat{t}_1, ..., \hat{t}_n)^T$ is a column vector of stochastic temperatures at the nodes of the thermal network; n is the number of nodes in the network with the exception of the reference node, which corresponds to zero temperature: $\hat{\bar{P}} = (\hat{P}_1, ..., \hat{P}_m)^T$ is a column vector of stochastic heat-emission intensities; m is the number of generalized edges of the graph representing the network; $\hat{\bar{t}}_c = (\hat{t}_{c1}, ..., \hat{t}_{cm})^T$ is a column vector of stochastic temperatures of air and a coolant; A is an n × m rectangular topological incidence matrix; $\hat{G} = [\hat{G}_{ii}]$, $i = 1, 2, ..., m$ is an m × m stochastic diagonal matrix of the thermal conductances of the generalized edges of the network graph; T denotes transposition; and, Λ denotes stochasticity. The elements of the stochastic matrix $\hat{G}$ are independent of one another and of the vectors $\hat{\bar{P}}$ and $\hat{\bar{t}}_c$.

We represent each element $\hat{G}_{ii}$ of the matrix $\hat{G}$ as a sum of its mathematical expectation $M_{G_{ii}}$ and the centered random quantity $\hat{G}^0_{ii}$. Then the matrix $\hat{G}$ can be written in the form $\hat{G} = M_G + \hat{G}^0$, where M_G and $\hat{G}^0$ are the diagonal matrix of mathematical expectation and a centered stochastic diagonal matrix, while the stochastic matrix $A\hat{G}A^T$ in Eq. (1) assumes the form

$$A\hat{G}A^{T} = U + \hat{S} = U(E + U^{-1}\hat{S}), \tag{2}$$

where $U = AM_GA^T$ and $\hat{S} = A\hat{G}^0A^T$ are determinate and stochastic n × n square matrices, and E is a diagonal unit matrix.

The solution of Eq. (1) with the transformed matrix $A\hat{G}A^T$ (2) can be expressed in an explicit form, if the stochastic matrix $E + U^{-1}\hat{S}$ is invertible. The matrix $E + U^{-1}\hat{S}$, however, is invertible, if there exists a matrix norm $\|\cdot\|$ such that $\|U^{-1}\hat{S}\| < 1$ [8]. In this case the desired column vector $\hat{\bar{t}}$ of stochastic temperatures will be determined by the following expresssion:

$$\hat{\bar{t}} = \hat{H}^{-1}U^{-1}A(\hat{\bar{P}} - M_G\hat{\bar{t}}_c - \hat{G}^0\hat{\bar{t}}_c), \tag{3}$$

where $\hat{H}^{-1} = (E + U^{-1}\hat{S})^{-1}$ is a stochastic inverse matrix.

Vector of Mathematical Expectation. We now find the vector of mathematical expectation $\bar{m}_t$ of the stochastic temperatures $\hat{\bar{t}}$. For this we apply the mathematical expectation operator M(·) to both sides of Eq. (3). Since the random thermal conductances are independent of the stochastic vector of powers $\hat{\bar{P}}$ and the stochastic vector of temperatures of the medium (coolant) $\hat{\bar{t}}_c$, we obtain

$$\bar{m}_t = M(\hat{\bar{t}}) = M(\hat{H}^{-1})U^{-1}A(\bar{m}_P - M_G\bar{m}_{t_c}) - M(\hat{H}^{-1}U^{-1}A\hat{G}^0)\bar{m}_{t_c}. \tag{4}$$

The difficulty in obtaining an analytic expression for the vector $\bar{m}_t$ is to find the inverse stochastic matrix $\hat{H}^{-1}$ and its mathematical expectation. This, however, can be done, since $\|U^{-1}\hat{S}\| < 1$, which makes it possible to represent the inverse stochastic matrix $\hat{H}^{-1}$ by a convergent power series [8]:

$$\hat{H}^{-1} = (E + U^{-1}\hat{S})^{-1} = \sum_{k=0}^{\infty}(-1)^k(U^{-1}\hat{S})^k. \tag{5}$$

We now determine the expressions for $M(\hat{H}^{-1})$ and $M(\hat{H}^{-1}U^{-1}A\hat{G}^0)$, appearing in Eq. (4). We introduce the matrices $B = U^{-1}A$ and $F = A^TB$, where $U^{-1}\hat{S} = B\hat{G}^0A^T$. Using the representation (5) and summing the matrix series obtained, we have

$$M(\hat{H}^{-1}) = \sum_{k=0}^{\infty}(-1)^k M(U^{-1}\hat{S})^k = \sum_{k=0}^{\infty}(-1)^k M(B\hat{G}^0A^T)^k = \\ = E - B\sum_{k=0}^{\infty}(-1)^k F_d^k M(\hat{G}^{0^{k+1}})A^T = E - BW_1A^T \tag{6}$$

if $\|F_d\hat{G}^0\| < 1$, where W_1 is a diagonal matrix with the elements

$$W_{1,ii} = M(\hat{G}^0_{ii}/(1 + F_{d,ii}\hat{G}^0_{ii})),\ i = 1,2,\ \dots,\ m; \quad (7)$$

F_d is a diagonal matrix whose elements $F_{d,ii}$, i = 1, 2, ..., m, are the diagonal elements of the matrix F. We note that since the matrix $F_d\hat{G}^0$ is diagonal, the condition $\|F_d\hat{G}^0\| < 1$ is equivalent to the condition $\max_i |F_{d,ii}\hat{G}^0_{ii}| < 1$.

Analogously to Eq. (6) we obtain

$$M(\hat{H}^{-1}U^{-1}A\hat{G}^0) = \sum_{k=0}^{\infty}(-1)^k M((U^{-1}\hat{S})^k U^{-1}A\hat{G}^0) = \\ = \sum_{k=0}^{\infty}(-1)^k M((B\hat{G}^0A^T)^k U^{-1}A\hat{G}^0) = B\sum_{k=1}^{\infty}(-1)^k F_d^k M(\hat{G}^{0k+1}) = BW_1 \quad (8)$$

with the conditions $\max_i |F_{d,ii}\hat{G}^0_{ii}| < 1$.

Substituting the expressions (7) and (8) into Eq. (4), we obtain finally an analytical expression for the vector of mathematical expectation of the stochastic temperatures

$$\bar{m}_t = B(E - W_1F)(\bar{m}_p - M_G\bar{m}_{t_c}) - BW_1\bar{m}_{t_c}. \quad (9)$$

The elements $W_{1,ii}$ of the matrix W_1 are, according to Eq. (7), the mathematical expectations of the expressions $\hat{G}^0_{ii}/(1 + F_{d,ii}\hat{G}^0_{ii})$ and are calculated from the formula

$$W_{1,ii} = \int_{G'_{ii}}^{G''_{ii}} f_i(G_{ii})\, G^0_{ii}/(1 + F_{d,ii}G^0_{ii})\, dG_{ii}, \quad (10)$$

where $f_i(G_{ii})$ is the distribution function of the centered random value of the i-th thermal conductance G^0_{ii}, i = 1, 2, ..., m. The distribution function $f_i(G_{ii})$ can be arbitrary, but it is truncated into the interval $[G'_{ii}, G''_{ii}]$.

Correlation Matrix. We now find the correlation matrix K_t of the stochastic temperatures $\hat{t}$. Usually it is more convenient to determine the matrix of the second initial moments Γ_t, with which the matrix K_t is related by the relation $K_t = \Gamma_t - \bar{m}_t\bar{m}_t^T$ [4].

We write the transpose of the matrix (3)

$$\hat{t}^T = (\hat{P} - M_G\hat{t}_c - \hat{G}^0\hat{t}_c)^T A^T U^{-1}(\hat{H}^T)^{-1}, \quad (11)$$

where $(\hat{H}^T)^{-1} = (E + \hat{S}^T U^{-1})^{-1}$.

Applying the mathematical expectation operator to the product $\hat{t}\hat{t}^T$, made up of the expressions (3) and (11), we obtain a matrix of second initial moments

$$\Gamma_t = M(\hat{t}\hat{t}^T) = M(\hat{H}^{-1}U^{-1}A(\hat{P} - M_G\hat{t}_c - \hat{G}^0\hat{t}_c) \times (\hat{P} - M_G\hat{t}_c - \hat{G}^0\hat{t}_c)^T A^T U^{-1}(\hat{H}^T)^{-1}) = \\ = M(\hat{H}^{-1}B\hat{Z}_1B^T(\hat{H}^T)^{-1}) - M(\hat{H}^{-1}B\hat{G}^0\hat{Z}_2B^T(\hat{H}^T)^{-1}) - M(\hat{H}^{-1}B\hat{Z}_2^T\hat{G}^0B^T(\hat{H}^T)^{-1}) + M(\hat{H}^{-1}B\hat{G}^0\hat{Z}_3\hat{G}^0B^T(\hat{H}^T)^{-1}) = \\ = \Gamma_1 - \Gamma_2 - \Gamma_3 + \Gamma_4, \quad (12)$$

where $\hat{Z}_1 = (\hat{P} - M_G\hat{t}_c)(\hat{P} - M_G\hat{t}_c)^T$, $\hat{Z}_2 = \hat{t}_c(\hat{P} - M_G\hat{t}_c)^T$ and $\hat{Z}_3 = \hat{t}_c\hat{t}_c^T$ are stochastic matrices; Γ_1, Γ_2, Γ_3, and Γ_4 are the mathematical expectations of the corresponding stochastic matrices in the expression (12). If the stohastic matrix $\hat{H}^{-1}$ is represented by the series (5), then the matrices Γ_1–Γ_4 in Eq. (12) can be put into the following form:

$$\Gamma_1 = B\cdot M\left(\sum_{i,j=0}^{\infty}(-1)^{i+j}(\hat{G}^0F)^i\hat{Z}_1(F\hat{G}^0)^j\right)B^T, \quad (13)$$

$$\Gamma_2 = B\cdot M\left(\sum_{i,j=0}^{\infty}(-1)^{i+j}(\hat{G}^0F)^i\hat{G}^0\hat{Z}_2(F\hat{G}^0)^j\right)B^T, \quad (14)$$

$$\Gamma_3 = B\cdot M\left(\sum_{i,j=0}^{\infty}(-1)^{i+j}(\hat{G}^0F)^i\hat{Z}_2^T\hat{G}^0(F\hat{G}^0)^j\right)B^T, \quad (15)$$

$$\Gamma_4 = B \cdot M \left(\sum_{i,j=0}^{\infty} (-1)^{i+j} (\hat{G}^0 F)^i \hat{G}^0 \hat{Z}_3 \hat{G}^0 (F\hat{G}^0)^j \right) B^{\mathrm{T}}. \tag{16}$$

Summing the series (13)-(16), substituting the expressions obtained for Γ_1-Γ_4 into Eq. (12), we obtain the final analytic expression for the matrix of second initial moments of the stochastic temperatures

$$\Gamma_t = B\,(M(\hat{Z}_1) - (M(\hat{Z}_1) F W_1 + W_1 F M(\hat{Z}_1)) + (F_d W_2 M(\hat{Z}_2) + M(\hat{Z}_2^{\mathrm{T}}) F_d W_2) + W_3 D)\, B^{\mathrm{T}} \tag{17}$$

under the condition $\max_i |F_{d,ii} \hat{G}^0_{ii}| < 1$, where W_2 is a diagonal matrix whose elements are

$$W_{2,ii} = M(\hat{G}^{0^2}_{ii} / (1 + F_{d,ii} \hat{G}^0_{ii})),\ i = 1, 2, \ldots, m; \tag{18}$$

W_3 is a diagonal matrix whose elements are

$$W_{3,ii} = M(\hat{G}^{0^2}_{ii} / (1 + F_{d,ii} \hat{G}^0_{ii})^2),\ i = 1, 2, \ldots, m; \tag{19}$$

D is a diagonal matrix whose elements D_{ii} have the form

$$D_{ii} = \sum_{q,k=1}^{m} M(\hat{Z}_{1,qk})\ F_{iq} F_{ki} + \sum_{k=1}^{m} M(\hat{Z}_{2,ik}) (F_{ik} + F_{ki}) + M(\hat{Z}_{3,ii}),\quad i = 1, 2, \ldots, m; \tag{20}$$

$M(\hat{Z}_1)$, $M(\hat{Z}_2)$, $M(\hat{Z}_3)$ are the mathematical expectations of the stochastic matrices $\hat{Z}_1$, $\hat{Z}_2$, $\hat{Z}_3$, i.e., the adjoint matrices of the second initial moments; $M(\hat{Z}_{1,qk})$, $M(\hat{Z}_{2,ik})$, $M(\hat{Z}_{3,ii})$ are the corresponding elements of the matrices $M(\hat{Z}_1)$, $M(\hat{Z}_2)$, $M(\hat{Z}_3)$. The elements $W_{2,ii}$ and $W_{3,ii}$ of the diagonal matrices W_2 and W_3 are calculated from formulas of the type (10).

The variances $\bar{\Theta}_t = (\Theta_1, \Theta_2, \ldots, \Theta_n)^{\mathrm{T}}$ of the stochastic temperatures of the technical object are equal to the diagonal elements of the correlation matrix K_t.

Thus we have derived analytical expressions which permit calculating exactly the mathematical expectations (9), the correlation matrix (17), and the variances of the stochastic temperatures at different points of technical objects in the static regime. The expressions obtained are valid, if $\max_i |F_{d,ii} \hat{G}^0_{ii}| < 1$. This condition is not strict, since in real technical objects the maximum spread of the random values of the thermal conductances is always less than their mathematical expectation. For the stochastic vectors $\hat{\hat{P}}$ and $\hat{t}$ it is necessary to know only the probabilistic characteristics; there is no need to know the distribution functions.

We now assess the efficiency of the proposed method as compared with the Monte Carlo method. For this we estimate the number of arithmetic operations which must be performed in order to determine the probabilistic charactersitics $\bar{m}_t$, K_t, and $\bar{\Theta}_t$. Since approximately n^3 operations must be performed in order to compute the product of two $n \times n$ matrices, and $2n^2/3$ operations must be performed in order to solve a system of n linear equations by Gauss' method, and $2n^3$ operations must be performed in order to invert an $n \times n$ matrix [9], the proposed method requires approximately $2n^2$ operations in order to determine the vector $\bar{m}_t$ and $9n^3$ operations in order to determine the matrix K_t and the vector $\bar{\Theta}_t$. At the same time, the Monte Carlo method requires approximately $2n^2N$ operations in order to determine the vectors $\bar{m}_t$ and $\bar{\Theta}_t$ and the matrix K_t, where N is the number of tests. We note that the actual machine time required will be somewhat larger because other operations must also be performed (readdressing, logical operations, data transfers, etc.), and also in the Monte Carlo method nN random numbers with prescribed distributions must be generated.

Example. We consider the stochastic temperature distribution in an electronic module containing a board containing four integrated circuits (IC) in a ceramic 16-output housing and two IC in a ceramic 40-output housing (see Fig. 1). The conditions of cooling correspond to forced ventilation. Radiation and heat conduction through the thickness of the board are neglected; the ends of the board are assumed to be thermally insulated. The thermal network (see Fig. 1) of the module contains 81 branches and 39 nodes (the numbers on the network are the branch numbers; the circled numbers are the numbers of the nodes corresponding to the

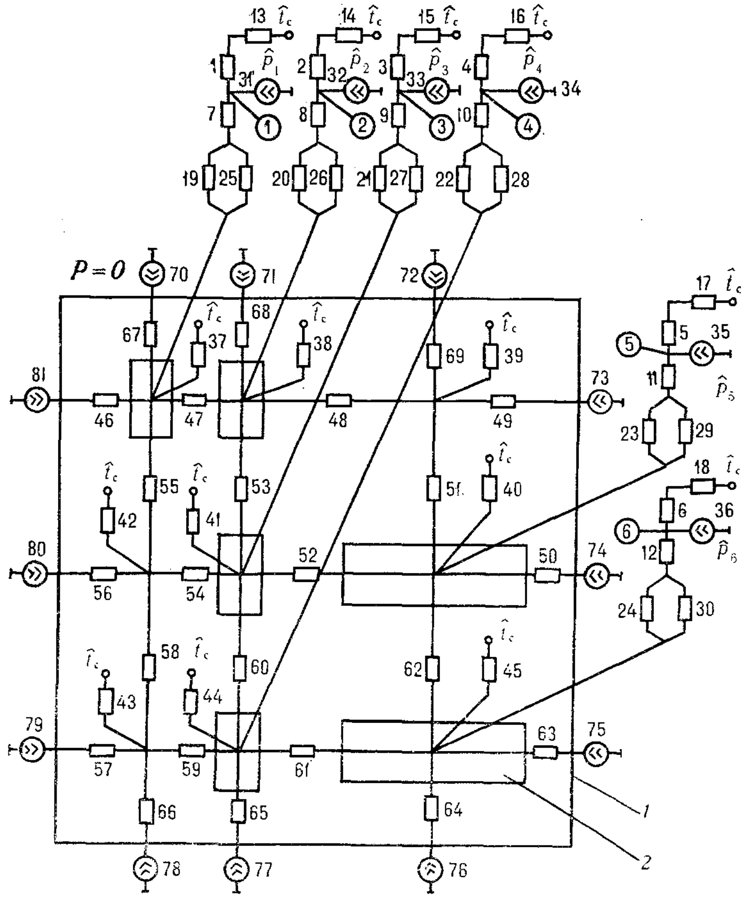

Fig. 1. Thermal network of the electronic module: 1) board; 2) integrated circuits placed on the board.

TABLE 1. Probabilistic Characteristics of Stochastic Temperatures of Crystals of IC Placed on the Board

Number of the IC on the board	Mathematical expectation m_t, °C	Variance Θ_t, °C²	Interval of possible values of stochastic temperatuer $\hat{t}$, °C, with confidence probability 0.95
1	83,03	14,88	75,47—90,60
2	82,76	14,82	75,22—90,30
3	81,46	12,75	74,46—88,46
4	81,38	31,41	70,39—92,36
5	98,66	15,90	90,84—106,48
6	99,58	21,69	90,45—108,71

crystals of the IC). The stochastic quantities are: the powers of the IC $\hat{P}_1$–$\hat{P}_6$ (edges 31-36); the coefficient of heat transfer into the medium α; the temperature of the medium $\hat{t}_c$; the thermal conductivities of the crystal IC-housing cover $\hat{\sigma}_1$–$\hat{\sigma}_6$ and of the crystal IC-housing base $\hat{\sigma}_7$–$\hat{\sigma}_{12}$, dissipation from the surface of the IC housing into the medium $\hat{\sigma}_{13}$–$\hat{\sigma}_{18}$, the IC-housing base board $\hat{\sigma}_{19}$–$\hat{\sigma}_{24}$, and dissipation from the surface of board free of IC into the medium $\hat{\sigma}_{37}$–$\hat{\sigma}_{45}$. The thermal conductances of the outputs of the housing $\hat{\sigma}_{25}$–$\hat{\sigma}_{30}$ and of the spreading of the heat flux over the board σ_{46}–σ_{69} are determinate quantities. The heat sources in the branches 70-81 are equal to zero and they model the adiabatic conditions on the ends of the board. The powers of the IC $\hat{P}_1$–$\hat{P}_6$ and the temperature of the medium $\hat{t}_c$ satisfy the distribution laws with the ratios $\varepsilon = \delta/m$ equal to $\varepsilon_{p1} + \varepsilon_{p2} = \varepsilon_{p3} = 23\%$, $\varepsilon_{p4} = 52.5\%$, $\varepsilon_{p5} = 17\%$, $\varepsilon_{p6} = 26\%$ and $\varepsilon_{t_c} = 48\%$, where δ and m are the range of possible values and the mathematical expectation of the stoachastic equantity. The stochastic thermal conductances $\hat{\sigma}_1$–$\hat{\sigma}_{18}$ and $\hat{\sigma}_{37}$–$\hat{\sigma}_{45}$ have a truncated normal distribution function while $\hat{\sigma}_{19}$–$\hat{\sigma}_{24}$

have a uniform distribution function with the ratios $\varepsilon_1 = \varepsilon_4 = \varepsilon_{19} = \varepsilon_{20} = \varepsilon_{39} = 44\%$, $\varepsilon_5 = 36\%$, $\varepsilon_6 = 21\%$, $\varepsilon_7 = \varepsilon_{10} = 53\%$, $\varepsilon_{11} = \varepsilon_{12} = 61\%$, $\varepsilon_{13} = \varepsilon_{16} = 46\%$, $\varepsilon_{17} = 13\%$, $\varepsilon_{18} = 42\%$, $\varepsilon_{23} = \varepsilon_{24} = 22\%$, $\varepsilon_{37} = \varepsilon_{38} = \varepsilon_{41} = 28\%$, $\varepsilon_{40} = \varepsilon_{44} = \varepsilon_{45} = 45\%$, $\varepsilon_{21} = \varepsilon_{22} = \varepsilon_{42} = \varepsilon_{43} = 31\%$. The norm $\|F_d\hat{G}^0\| = 0.523$ and corresponds to the thermal conductance $\hat{\sigma}_{12}$.

The vectors of mathematical expectation $\bar{m}_t$ and variance $\bar{\theta}_t$ and the correlation matrix K_t of the stochastic temperatures at 39 nodes of the thermal network were calculated using the expressions of the method (9) and (17) on an IBM PC/AT-386 personal computer. The results are presented in Table 1 for the probabilistic charactersitics of the stochastic temperatures of the crystals of the integrated circuits (nodes 1-6 of the thermal network), which are most important for design purposes. The last column in Table 1 shows that the temperature of 95% of the crystals of the IC (in this example) will fall within the intervals given.

For comparison we calculated the probabilistic characteristics by the Monte Carlo method. In order to obtain results which differ from the exact results (obtained by the proposed method) with a relative error of 5%, 500 tests and 1 h 41.6 min of machine time were required. The method proposed in the present paper required 1.4 min of machine time.

Conlusions. The proposed method permits determining the mathematical expectation, the correlation matrix, and the variance of the stochastic temperature distribution in technical objects of arbitrary complexity, which are modelled by equivalent electric circuits. Applications of the method to different technical objects showed that it is significantly more efficient than the Monte Carlo method.

NOTATION

$\hat{t}$, vector of stochastic temperatures; $\hat{G}$, stochastic matrix of thermal conductances; $\hat{P}$, $\hat{t}_c$, vectors of stochastic powers and temperatures of the medium or coolant; A, incidence matrix; $M(\cdot)$, mathematical expectation operator; K_t, correlation matrix; $\bar{m}_t$, $\bar{\theta}_t$, vectors of mathematical expectations and variances of stochastic temperatures; and, Γ_t, matrix of second initial moments.

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